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Diagonalization of the System of Static Lamé Equations of Isotropic Linear Elasticity

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Abstract—We find a simplest representation for the general solution to the system of the static Lamé equations of isotropic linear elasticity in the form of a linear combination of the first derivatives of three functions that satisfy three independent harmonic equations. The representation depends on 12 free parameters choosing which it is possible to obtain various representations of the general solution and simplify the boundary value conditions for the solution of boundary value problems as well as the representation of the general solution for dynamic Lamé equations. The system of Lamé equations diagonalizes; i.e., it is reduced to the solution of three independent harmonic equations. The representation implies three conservation laws and some formula for producing new solutions which makes it possible, given a solution, to find new solutions to the static Lamé equations by derivations. In the two-dimensional case of a plane deformation, the so-found solution immediately implies the Kolosov—Muskhelishvili representation for shifts by means of two analytic functions of complex variable. Two examples are given of applications of the proposed method of diagonalization of the two-dimensional elliptic systems.

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The article develops the approach that was suggested in [1, 2]. For an isotropic material and without influence of the volume forces, the system of static Lamé equations in displacements of linear elasticity theory has the form

$$[(\lambda+\mu)\partial_{ij}+\mu\partial_{kk}\delta_{ij}]u_j=0, \qquad i,j,k=1,2,3, \tag{1}$$

where λ and μ are the Lamé constants; δ_{ij} is the Kronecker symbol; u_j is the displacement vector; ∂_j is the derivative with respect to the coordinate x_j, $j=1,2,3$; $\partial_{ij}=\partial_i\partial_j$ is the second derivative with respect to x_i and x_j; and repeated indices imply summation.

For system (1), there exist different representations of the general solution by means of the resolving potentials [3, 4]. In the present article, we obtain a simplest representation for the general solution to (1) in the form of a linear combination of the first derivatives of three functions satisfying three independent harmonic equations. This representation looks as follows:

$$u_j=(\alpha\partial_j+\varepsilon_{jms}a_m\partial_s)\varphi_1+(\beta\partial_j+\varepsilon_{jms}b_m\partial_s)\varphi_2+(\gamma\partial_j+\varepsilon_{jms}c_m\partial_s)\varphi_3, \qquad j=1,2,3, \tag{2}$$

where the functions φ_1, φ_2, and φ_3 meet the equations

$$\partial_{kk}\varphi_1=f_1, \qquad \partial_{kk}\varphi_2=f_2, \qquad \partial_{kk}\varphi_3=f_3, \tag{3}$$

and the functions f_1, f_2, and f_3 satisfy

$$[(\lambda+2\mu)\alpha\partial_i+\mu\varepsilon_{ims}a_m\partial_s]f_1+[(\lambda+2\mu)\beta\partial_i+\mu\varepsilon_{ims}b_m\partial_s]f_2 + [(\lambda+2\mu)\gamma\partial_i+\mu\varepsilon_{ims}c_m\partial_s]f_3=0, \qquad i=1,2,3. \tag{4}$$

Here α, β, γ, a_m, b_m, and c_m with $m=1,2,3$ are arbitrary parameters such that the three quantities $\varepsilon_{snp}(\alpha b_nc_p+\beta c_na_p+\gamma a_nb_p)$, $s=1,2,3$, cannot vanish simultaneously. Here ε_{jms} is the antisymmetric Levi—Civita tensor.

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Choosing the free parameters, we can obtain various versions of the representations of the general solution (2)–(4) and simplify the boundary conditions. For $a_m = 0$ with $m = 1, 2, 3$, $\beta = 0$, and $\gamma = 0$, (2)–(4) imply the representation [1] of the general solution for the dynamic equations (1).

Denoting by A, C, D, and B the matrices of the operators in (1)–(4), where D is a diagonal matrix, we obtain $AC = BD$. Then the general solution (2)–(4) to (1) ($Au = 0$) has the form [5]

$$u = C\varphi, \qquad D\varphi = f, \qquad Bf = 0. \tag{5}$$

The matrix of the operator A of (1) is as follows:

$$A = \begin{bmatrix} (\lambda+\mu)\partial_{11} + \mu\partial_{kk} & (\lambda+\mu)\partial_{12} & (\lambda+\mu)\partial_{13} \\ (\lambda+\mu)\partial_{21} & (\lambda+\mu)\partial_{22} + \mu\partial_{kk} & (\lambda+\mu)\partial_{23} \\ (\lambda+\mu)\partial_{31} & (\lambda+\mu)\partial_{32} & (\lambda+\mu)\partial_{33} + \mu\partial_{kk} \end{bmatrix}. \tag{6}$$

The determinant of the matrix (6) is equal to

$$|A| = (\lambda+2\mu)\mu^2(\partial_{kk})^3 = D_1 D_2^2, \qquad D_1 = (\lambda+2\mu)\partial_{kk}, \qquad D_2 = D_3 = \mu\partial_{kk}, \tag{7}$$

while the matrix D in the solution (5) is diagonal:

$$D = \operatorname{diag}(D_1, D_2, D_3) = \begin{bmatrix} (\lambda+2\mu)\partial_{kk} & 0 & 0 \\ 0 & \mu\partial_{kk} & 0 \\ 0 & 0 & \mu\partial_{kk} \end{bmatrix}. \tag{8}$$

From (7) we see that the Lamé system (1) is elliptic if $|A| > 0$ for all real values of the symbols ∂_k, $\partial_{kk} \neq 0$; i.e., the constants satisfy

$$\lambda + 2\mu > 0, \qquad \mu \neq 0; \tag{9}$$

system (1) is strongly elliptic if [6]

$$\lambda + 2\mu > 0, \qquad \mu > 0. \tag{10}$$

The specific deformation energy $2\Phi = \lambda\varepsilon_{ii}\varepsilon_{kk} + 2\mu\varepsilon_{ij}\varepsilon_{ij}$, $\varepsilon_{ij} = (\partial_i u_j + \partial_j u_i)/2$ is a positive definite quadratic form if [3]

$$3\lambda + 2\mu > 0, \qquad \mu > 0. \tag{11}$$

The domains of admissible values for the constants λ and μ defined by (9)–(11) are shown in the figure. The domain of ellipticity for (9) consists of three parts I, II, and III; moreover, λ and μ can be negative. The domain of elasticity for (11) is part II; here λ can be negative but $\mu > 0$. The domain of strong ellipticity for (10) includes parts II and III. If the constants λ and μ belong to parts I and III then equations (1) are no longer equations of elasticity theory because conditions (11) are not fulfilled.

Suppose that the matrices C and B have the form $C_{jq} = \alpha_{jpq}\partial_p$ and $B_{ip} = \beta_{isp}\partial_s$, where α_{jpq} and β_{isp} are some coefficients. Choosing the displacements in the form

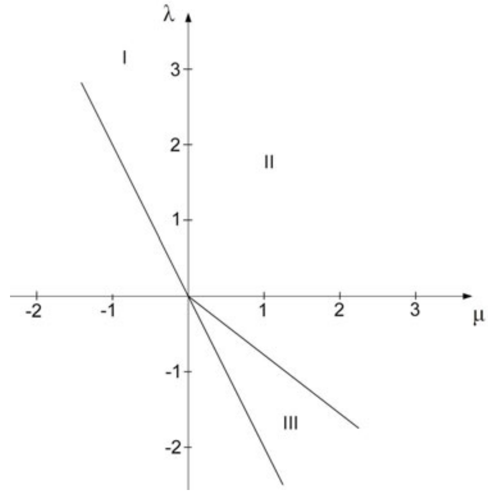
$$u_j = C_{jq}\varphi_q = \alpha_{jp1}\partial_p\varphi_1 + \alpha_{jp2}\partial_p\varphi_2 + \alpha_{jp3}\partial_p\varphi_3, \qquad j, p = 1, 2, 3,$$

and requiring the fulfillment of the relation

$$AC = BD, \qquad A_{ij}C_{jq} = B_{ip}D_{pq}, \tag{12}$$

we obtain the matrices

$$C = \begin{bmatrix} \alpha\partial_1 + a_2\partial_3 - a_3\partial_2 & \beta\partial_1 + b_2\partial_3 - b_3\partial_2 & \gamma\partial_1 + c_2\partial_3 - c_3\partial_2 \\ \alpha\partial_2 + a_3\partial_1 - a_1\partial_3 & \beta\partial_2 + b_3\partial_1 - b_1\partial_3 & \gamma\partial_2 + c_3\partial_1 - c_1\partial_3 \\ \alpha\partial_3 + a_1\partial_2 - a_2\partial_1 & \beta\partial_3 + b_1\partial_2 - b_2\partial_1 & \gamma\partial_3 + c_1\partial_2 - c_2\partial_1 \end{bmatrix}, \tag{13}$$



Domain of admissible values for the constants λ and μ defined by (9)–(11)

$$B = \begin{bmatrix} \alpha\partial_1 + \frac{\mu}{\lambda+2\mu}(a_2\partial_3 - a_3\partial_2) & \frac{\lambda+2\mu}{\mu}\beta\partial_1 + b_2\partial_3 - b_3\partial_2 & \frac{\lambda+2\mu}{\mu}\gamma\partial_1 + c_2\partial_3 - c_3\partial_2 \\ \alpha\partial_2 + \frac{\mu}{\lambda+2\mu}(a_3\partial_1 - a_1\partial_3) & \frac{\lambda+2\mu}{\mu}\beta\partial_2 + b_3\partial_1 - b_1\partial_3 & \frac{\lambda+2\mu}{\mu}\gamma\partial_2 + c_3\partial_1 - c_1\partial_3 \\ \alpha\partial_3 + \frac{\mu}{\lambda+2\mu}(a_1\partial_2 - a_2\partial_1) & \frac{\lambda+2\mu}{\mu}\beta\partial_3 + b_1\partial_2 - b_2\partial_1 & \frac{\lambda+2\mu}{\mu}\gamma\partial_3 + c_1\partial_2 - c_2\partial_1, \end{bmatrix},$$

or, briefly,

$$C_{jq} = [\alpha\partial_j + \varepsilon_{jms}a_m\partial_s,\ \beta\partial_j + \varepsilon_{jms}b_m\partial_s,\ \gamma\partial_j + \varepsilon_{jms}c_m\partial_s],$$
$$B_{ip} = \left[\alpha\partial_i + \frac{\mu}{\lambda+2\mu}\varepsilon_{ims}a_m\partial_s,\ \frac{\lambda+2\mu}{\mu}\beta\partial_i + \varepsilon_{ims}b_m\partial_s,\ \frac{\lambda+2\mu}{\mu}\gamma\partial_i + \varepsilon_{ims}c_m\partial_s\right],$$

where α, β, γ, a_m, b_m, and c_m with $m = 1, 2, 3$ are free parameters.

A straightforward check shows that (12) holds for the matrices (6), (8), and (13). The determinants of (6) and (8) obviously coincide: $|A| = |D| = D_1D_2^2$. The determinant of the matrix C in (13) is equal to

$$\begin{aligned} |C| &= [\varepsilon_{snp}(\alpha b_nc_p + \beta c_na_p + \gamma a_nb_p)\partial_s]\partial_{kk} \\ &= \{[\alpha(b_2c_3 - b_3c_2) + \beta(c_2a_3 - c_3a_2) + \gamma(a_2b_3 - a_3b_2)]\partial_1 \\ &\quad + [\alpha(b_3c_1 - b_1c_3) + \beta(c_3a_1 - c_1a_3) + \gamma(a_3b_1 - a_1b_3)]\partial_2 \\ &\quad + [\alpha(b_1c_2 - b_2c_1) + \beta(c_1a_2 - c_2a_1) + \gamma(a_1b_2 - a_2b_1)]\partial_3\}\partial_{kk}. \end{aligned} \quad (14)$$

The matrix B in (13) has the same structure as C. To obtain $|B|$, we must replace in (14) the coefficients

$$a_m \to \frac{\mu}{\lambda+2\mu}a_m, \qquad \beta \to \frac{\lambda+2\mu}{\mu}\beta, \qquad \gamma \to \frac{\lambda+2\mu}{\mu}\gamma;$$

then

$$|B| = \left[\varepsilon_{snp}\left(\alpha b_nc_p + \frac{\lambda+2\mu}{\mu}\beta c_n\frac{\mu}{\lambda+2\mu}a_p + \frac{\lambda+2\mu}{\mu}\gamma\frac{\mu}{\lambda+2\mu}a_nb_p\right)\partial_s\right]\partial_{kk} = |C|,$$

i.e., the determinants of C and B coincide.

The free parameters in (13) must be such that the determinant $|C|$ in (14) is nonzero; i.e., the coefficients $\varepsilon_{snp}(\alpha b_nc_p + \beta c_na_p + \gamma a_nb_p)$, $s = 1, 2, 3$, at ∂_s must not vanish simultaneously.

For the dynamic equations (1), the matrices C and B in (13) must coincide, then

$$a_m = \frac{\mu}{\lambda+2\mu}a_m, \qquad \beta = \frac{\lambda+2\mu}{\mu}\beta, \qquad \gamma = \frac{\lambda+2\mu}{\mu}\gamma;$$

moreover, $a_m = 0$ for $m = 1, 2, 3$, $\beta = 0$, $\gamma = 0$, $C_{jp} = B_{jp} = [\alpha\partial_j,\ \varepsilon_{jms}b_m\partial_s,\ \varepsilon_{jms}c_m\partial_s]$; i.e., the matrices are similar to those obtained earlier for dynamics [1].

Since the matrix D in (8) looks as $\operatorname{diag}(\lambda + 2\mu,\ \mu,\ \mu)\operatorname{diag}(\partial_{kk},\ \partial_{kk},\ \partial_{kk})$, in the product BD, the factor $\operatorname{diag}(\lambda + 2\mu,\ \mu,\ \mu)$ can be attached to B; then B takes the form (4), and in (3) $D = \operatorname{diag}(\partial_{kk},\ \partial_{kk},\ \partial_{kk})$.

Thus, granted (6), (8), (12), and (13), the general solution (2)–(4) to (1) in the form (5) looks as follows:

$$\begin{aligned} u_1 &= (\alpha\partial_1 + a_2\partial_3 - a_3\partial_2)\varphi_1 + (\beta\partial_1 + b_2\partial_3 - b_3\partial_2)\varphi_2 + (\gamma\partial_1 + c_2\partial_3 - c_3\partial_2)\varphi_3, \\ u_2 &= (\alpha\partial_2 + a_3\partial_1 - a_1\partial_3)\varphi_1 + (\beta\partial_2 + b_3\partial_1 - b_1\partial_3)\varphi_2 + (\gamma\partial_2 + c_3\partial_1 - c_1\partial_3)\varphi_3, \\ u_3 &= (\alpha\partial_3 + a_1\partial_2 - a_2\partial_1)\varphi_1 + (\beta\partial_3 + b_1\partial_2 - b_2\partial_1)\varphi_2 + (\gamma\partial_3 + c_1\partial_2 - c_2\partial_1)\varphi_3, \end{aligned} \tag{15}$$

where the functions φ_1, φ_2, and φ_3 satisfy three independent equations (3) and the functions f_1, f_2, and f_3 enjoy

$$\begin{aligned} &[(\lambda + 2\mu)\alpha\partial_1 + \mu(a_2\partial_3 - a_3\partial_2)]f_1 + [(\lambda + 2\mu)\beta\partial_1 + \mu(b_2\partial_3 - b_3\partial_2)]f_2 \\ &\qquad + [(\lambda + 2\mu)\gamma\partial_1 + \mu(c_2\partial_3 - c_3\partial_2)]f_3 = 0, \\ &[(\lambda + 2\mu)\alpha\partial_2 + \mu(a_3\partial_1 - a_1\partial_3)]f_1 + [(\lambda + 2\mu)\beta\partial_2 + \mu(b_3\partial_1 - b_1\partial_3)]f_2 \\ &\qquad + [(\lambda + 2\mu)\gamma\partial_2 + \mu(c_3\partial_1 - c_1\partial_3)]f_3 = 0, \\ &[(\lambda + 2\mu)\alpha\partial_3 + \mu(a_1\partial_2 - a_2\partial_1)]f_1 + [(\lambda + 2\mu)\beta\partial_3 + \mu(b_1\partial_2 - b_2\partial_1)]f_2 \\ &\qquad + [(\lambda + 2\mu)\gamma\partial_3 + \mu(c_1\partial_2 - c_2\partial_1)]f_3 = 0. \end{aligned} \tag{16}$$

The strains $\sigma_{ij} = \sigma_{ji}$ are defined via the displacements by Hooke's law

$$\sigma_{ij} = \lambda\delta_{ij}\partial_k u_k + \mu(\partial_i u_j + \partial_j u_i);$$

moreover, with account taken of (2) and (15), we infer

$$\begin{aligned} \sigma_{ij} = {} & [\alpha(\lambda\delta_{ij}\partial_{kk} + 2\mu\partial_{ij}) + \mu(\varepsilon_{ims}\partial_j + \varepsilon_{jms}\partial_i)a_m\partial_s]\varphi_1 + [\beta(\lambda\delta_{ij}\partial_{kk} + 2\mu\partial_{ij}) \\ & + \mu(\varepsilon_{ims}\partial_j + \varepsilon_{jms}\partial_i)b_m\partial_s]\varphi_2 + [\gamma(\lambda\delta_{ij}\partial_{kk} + 2\mu\partial_{ij}) + \mu(\varepsilon_{ims}\partial_j + \varepsilon_{jms}\partial_i)c_m\partial_s]\varphi_3. \end{aligned}$$

Relations (12) imply

$$C'A = DB', \qquad C_{ip}A_{ij} = D_{pi}B_{ji}, \tag{17}$$

where the prime stands for matrix transposition. If $A\tilde{u} = 0$ then we obtain from (17) that $C'A\tilde{u} = DB'\tilde{u} = D\varphi = 0$; i.e., $\varphi = B'\tilde{u}$ satisfy three separate harmonic equations. Granted (12) and (17), the formulas $u = C\varphi$, $\varphi = B'\tilde{u}$, and $A\tilde{u} = 0$ take the solutions to the equations $Au = 0$ and $D\varphi = 0$ to each other. In general, there is no one-to-one correspondence between u and φ though A and D are equivalent as matrices since C and B are nondegenerate square matrices. Therefore, to avoid the loss of a part of the solutions, in the general solution (5) or (15), (3), (16), we take account of the functions f that are the kernel of B.

Granted (13), the functions $\varphi_p = B_{ip}\tilde{u}_i$ take the form

$$\varphi_1 = [(\lambda + 2\mu)\alpha\partial_i + \mu\varepsilon_{ims}a_m\partial_s]\tilde{u}_i, \qquad \varphi_2 = [(\lambda + 2\mu)\beta\partial_i + \mu\varepsilon_{ims}b_m\partial_s]\tilde{u}_i,$$
$$\varphi_3 = [(\lambda + 2\mu)\gamma\partial_i + \mu\varepsilon_{ims}c_m\partial_s]\tilde{u}_i.$$

The expression $u = CB'\tilde{u}$ is a formula for producing new solutions; i.e., if $A\tilde{u} = 0$ then (12) and (17) imply that $u = CB'\tilde{u}$ is a new solution:

$$Au = ACB'\tilde{u} = BDB'\tilde{u} = BC'A\tilde{u} = 0.$$

Hence, $Q = CB'$ is the symmetry (recursion) operator [5, 7]; in addition,

$$\begin{aligned} Q_{ij} &= C_{ip}B'_{pj} = C_{ip}B_{jp} = C_{i1}B_{j1} + C_{i2}B_{j2} + C_{i3}B_{j3} \\ &= (\alpha\partial_i + \varepsilon_{ims}a_m\partial_s)[(\lambda + 2\mu)\alpha\partial_j + \mu\varepsilon_{jnr}a_n\partial_r] + (\beta\partial_i + \varepsilon_{ims}a_m\partial_s)[(\lambda + 2\mu)\beta\partial_j + \mu\varepsilon_{jnr}a_n\partial_r] \\ &\qquad + (\gamma\partial_i + \varepsilon_{ims}a_m\partial_s)[(\lambda + 2\mu)\gamma\partial_j + \mu\varepsilon_{jnr}a_n\partial_r], \end{aligned}$$

or, involving (13), we infer

$$Q_{11} = (\lambda+2\mu)a_{1k}a_{1k}\partial_{11} - (\lambda+3\mu)a_{1k}a_{4k}\partial_{12} + (\lambda+3\mu)a_{1k}a_{3k}\partial_{13} + \mu a_{4k}a_{4k}\partial_{22} - 2\mu a_{3k}a_{4k}\partial_{23} + \mu a_{3k}a_{3k}\partial_{33},$$

$$Q_{21} = (\lambda+2\mu)a_{1k}a_{4k}\partial_{11} + [(\lambda+2\mu)a_{1k}a_{1k} - \mu a_{4k}a_{4k}]\partial_{12} + [-(\lambda+2\mu)a_{1k}a_{2k} + \mu a_{3k}a_{4k}]\partial_{13} - \mu a_{1k}a_{4k}\partial_{22} + \mu(a_{1k}a_{3k} + a_{2k}a_{4k})\partial_{23} - \mu a_{2k}a_{3k}\partial_{33},$$

$$Q_{31} = -(\lambda+2\mu)a_{1k}a_{3k}\partial_{11} + [(\lambda+2\mu)a_{1k}a_{2k} + \mu a_{3k}a_{4k}]\partial_{12} + [(\lambda+2\mu)a_{1k}a_{1k} - \mu a_{3k}a_{3k}]\partial_{13} - \mu a_{2k}a_{4k}\partial_{22} + \mu(-a_{1k}a_{4k} + a_{2k}a_{3k})\partial_{23} + \mu a_{1k}a_{3k}\partial_{33};$$

$$Q_{12} = \mu a_{1k}a_{4k}\partial_{11} + [(\lambda+2\mu)a_{1k}a_{1k} - \mu a_{4k}a_{4k}]\partial_{12} + \mu(-a_{1k}a_{2k} + a_{3k}a_{4k})\partial_{13} - (\lambda+2\mu)a_{1k}a_{4k}\partial_{22} + [(\lambda+2\mu)a_{1k}a_{3k} + \mu a_{2k}a_{4k}]\partial_{23} - \mu a_{2k}a_{3k}\partial_{33};$$

$$Q_{22} = \mu a_{4k}a_{4k}\partial_{11} + (\lambda+3\mu)a_{1k}a_{4k}\partial_{12} - 2\mu a_{2k}a_{4k}\partial_{13} + (\lambda+2\mu)a_{1k}a_{1k}\partial_{22} - (\lambda+3\mu)a_{1k}a_{2k}\partial_{23} + \mu a_{2k}a_{2k}\partial_{33};$$

$$Q_{32} = -\mu a_{3k}a_{4k}\partial_{11} + [-(\lambda+2\mu)a_{1k}a_{3k} + \mu a_{2k}a_{4k}]\partial_{12} + \mu(a_{1k}a_{4k} + a_{2k}a_{3k})\partial_{13} + (\lambda+2\mu)a_{1k}a_{2k}\partial_{22} + [(\lambda+2\mu)a_{1k}a_{1k} - \mu a_{2k}a_{2k})\partial_{23} - \mu a_{1k}a_{2k}\partial_{33};$$

$$Q_{13} = -\mu a_{1k}a_{3k}\partial_{11} + \mu(a_{1k}a_{2k} + a_{3k}a_{4k})\partial_{12} + [(\lambda+2\mu)a_{1k}a_{1k} - \mu a_{3k}a_{3k}]\partial_{13} - \mu a_{2k}a_{4k}\partial_{22} + [-(\lambda+2\mu)a_{1k}a_{4k} + \mu a_{2k}a_{3k}]\partial_{23} + (\lambda+2\mu)a_{1k}a_{3k}\partial_{33};$$

$$Q_{23} = -\mu a_{3k}a_{4k}\partial_{11} + \mu(-a_{1k}a_{3k} + a_{2k}a_{4k})\partial_{12} + [(\lambda+2\mu)a_{1k}a_{4k} + \mu a_{2k}a_{3k}]\partial_{13} + \mu a_{1k}a_{2k}\partial_{22} + [(\lambda+2\mu)a_{1k}a_{1k} - \mu a_{2k}a_{2k}]\partial_{23} - (\lambda+2\mu)a_{1k}a_{2k}\partial_{33};$$

$$Q_{33} = \mu a_{3k}a_{3k}\partial_{11} - 2\mu a_{2k}a_{3k}\partial_{12} - (\lambda+3\mu)a_{1k}a_{3k}\partial_{13} + \mu a_{2k}a_{2k}\partial_{22} + (\lambda+3\mu)a_{1k}a_{2k}\partial_{23} + (\lambda+2\mu)a_{1k}a_{1k}\partial_{33},$$

where we put

$$a_{ik} = \begin{bmatrix} \alpha & \beta & \gamma \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, \tag{18}$$

$$a_{ik}a_{jk} = a_{i1}a_{j1} + a_{i2}a_{j2} + a_{i3}a_{j3}, \qquad i,j = \overline{1,4}.$$

If u_j is a solution to (1) then from (17) it follows that

$$C_{i1}A_{ij}u_j = D_{11}B_{j1}u_j = \partial_{kk}[\alpha(\lambda+2\mu)\partial_j + \mu\varepsilon_{jms}a_m\partial_s]u_j = \partial_s\partial_{kk}[\alpha(\lambda+2\mu)u_s - \mu\varepsilon_{smj}a_m u_j] = 0,$$

$$C_{i2}A_{ij}u_j = D_{22}B_{j2}u_j = \partial_{kk}[\beta(\lambda+2\mu)\partial_j + \mu\varepsilon_{jms}b_m\partial_s]u_j = \partial_s\partial_{kk}[\beta(\lambda+2\mu)u_s - \mu\varepsilon_{smj}b_m u_j] = 0, \tag{19}$$

$$C_{i3}A_{ij}u_j = D_{33}B_{j3}u_j = \partial_{kk}[\gamma(\lambda+2\mu)\partial_j + \mu\varepsilon_{jms}c_m\partial_s]u_j = \partial_s\partial_{kk}[\gamma(\lambda+2\mu)u_s - \mu\varepsilon_{smj}c_m u_j] = 0.$$

All three equations in (19) are similar and have the form of the conservation law $\partial_s A_s = 0$ [7, 8], where the conserved currents are equal.

$$A_s = \partial_{kk}[\alpha(\lambda + 2\mu)u_s - \mu\varepsilon_{smj}a_m u_j], \qquad s = 1, 2, 3,$$
$$B_s = \partial_{kk}[\beta(\lambda + 2\mu)u_s - \mu\varepsilon_{smj}b_m u_j], \qquad C_s = \partial_{kk}[\gamma(\lambda + 2\mu)u_s - \mu\varepsilon_{smj}c_m u_j].$$

Rewrite (2) as

$$u_j = \partial_j(\alpha\varphi_1 + \beta\varphi_2 + \gamma\varphi_3) - \varepsilon_{jsm}\partial_s(a_m\varphi_1 + b_m\varphi_2 + c_m\varphi_3), \qquad j = 1, 2, 3. \tag{20}$$

If in (20) we put $\alpha\varphi_1 + \beta\varphi_2 + \gamma\varphi_3 = \varphi$ and $-(a_m\varphi_1 + b_m\varphi_2 + c_m\varphi_3) = \psi_m$ for $m = 1, 2, 3$ then (20) coincides in its form with the Kelvin–Lamé representation [4, 5].

Rewrite (4) in the form

$$\partial_i[(\lambda + 2\mu)(\alpha f_1 + \beta f_2 + \gamma f_3)] - \varepsilon_{ism}\partial_s[\mu(a_m f_1 + b_m f_2 + c_m f_3)] = 0, \qquad i = 1, 2, 3. \tag{21}$$

If in (21) we put

$$(\lambda + 2\mu)(\alpha f_1 + \beta f_2 + \gamma f_3) = g, \qquad -\mu(a_m f_1 + b_m f_2 + c_m f_3) = g_m, \qquad m = 1, 2, 3, \tag{22}$$

then (21) becomes

$$\partial_i g + \varepsilon_{ism}\partial_s g_m = 0, \qquad i = 1, 2, 3; \tag{23}$$

moreover, (23) coincides in its form with the equations for the kernel of C in the Kelvin–Lamé representation (20) [5].

In (22), there are four equations for three functions f_i. For the solvability of this system, the rank of the matrix a_{ik} in (18) must be three, and the determinant of the extended matrix

$$b_{ik} = \begin{bmatrix} \alpha & \beta & \gamma & g/(\lambda + 2\mu) \\ a_1 & b_1 & c_1 & -g_1/\mu \\ a_2 & b_2 & c_2 & -g_2/\mu \\ a_3 & b_3 & c_3 & -g_3/\mu \end{bmatrix} \tag{24}$$

must be zero. Find the algebraic complements to the elements of the last column in (24):

$$m = -\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \qquad m_1 = \begin{vmatrix} \alpha & \beta & \gamma \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \qquad m_2 = -\begin{vmatrix} \alpha & \beta & \gamma \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}, \qquad m_3 = \begin{vmatrix} \alpha & \beta & \gamma \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}, \tag{25}$$

where

$$m = -(a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2),$$
$$m_1 = \alpha(b_2c_3 - b_3c_2) + \beta(c_2a_3 - c_3a_2) + \gamma(a_2b_3 - a_3b_2),$$
$$m_2 = \alpha(b_3c_1 - b_1c_3) + \beta(c_3a_1 - c_1a_3) + \gamma(a_3b_1 - a_1b_3),$$
$$m_3 = \alpha(b_1c_2 - b_2c_1) + \beta(c_1a_2 - c_2a_1) + \gamma(a_1b_2 - a_2b_1).$$

In (25), the quantities m_i coincide with the coefficients at ∂_s in the determinant $|C|$ in (14) and $m_1^2 + m_2^2 + m_3^2 \neq 0$. This inequality means that the rank of the matrix a_{ik} in (18) is equal to three; moreover, granted (25), the compatibility condition for (22) takes the form

$$\mu m g - (\lambda + 2\mu)(m_1g_1 + m_2g_2 + m_3g_3) = 0. \tag{26}$$

We must add (23) to (26): The matrix of system (26), (23) is as follows:

$$M = \begin{bmatrix} \mu m & -(\lambda + 2\mu)m_1 & -(\lambda + 2\mu)m_2 & -(\lambda + 2\mu)m_3 \\ \partial_1 & 0 & -\partial_3 & \partial_2 \\ \partial_2 & \partial_3 & 0 & -\partial_1 \\ \partial_3 & -\partial_2 & \partial_1 & 0 \end{bmatrix}.$$

The determinant of this matrix

$$|M| = (\lambda + 2\mu)(m_1\partial_1 + m_2\partial_2 + m_3\partial_3)\partial_{kk} = (\lambda + 2\mu)|C| \neq 0$$

is proportional to the determinant $|C|$ in (14). Hence, the solution of (26), (23) is equivalent to the solution of (4) or (16).

The Kelvin–Lamé representation of the form (20) is applied in dynamic problems [4], and it is unclear how to use it in the static case. Though this representation satisfies $AC = CD$ [5], the matrices A and D cannot be assumed equivalent since A is of third order, D is of fourth order, while C is a nonsquare matrix of size 3×4. In algebra, some matrices A and D are regarded as equivalent if $AC = BD$, where C and B are nondegenerate square matrices. The equivalence conditions are not fulfilled for the Kelvin–Lamé representation [4, 5]. In the representation (2)–(4) or (15), (3), (16), all matrices are square (or order three) and nondegenerate; i.e., the equivalence is fulfilled.

Consider the case of a plane deformation for which u_1 and u_2 are independent of x_3 and $u_3 = 0$. Then, instead of (6)–(8), we infer

$$A = \begin{bmatrix} (\lambda + 2\mu)\partial_{11} + \mu\partial_{22} & (\lambda + \mu)\partial_{12} \\ (\lambda + \mu)\partial_{21} & \mu\partial_{11} + (\lambda + \mu)\partial_{22} \end{bmatrix}, \tag{27}$$

$$|A| = (\lambda + 2\mu)\mu(\partial_{11} + \partial_{22})^2 = D_1D_2, \qquad D_1 = (\lambda + 2\mu)(\partial_{11} + \partial_{22}), \qquad D_2 = \mu(\partial_{11} + \partial_{22});$$

$$D = \operatorname{diag}(D_1, D_2) = \begin{bmatrix} (\lambda + 2\mu)(\partial_{11} + \partial_{22}) & 0 \\ 0 & \mu(\partial_{11} + \partial_{22}) \end{bmatrix}, \qquad |D| = |A|. \tag{28}$$

The system of equations for u_1, u_2 is elliptic if $(\lambda + 2\mu)\mu > 0$. The matrices (13) take the form

$$C = \begin{bmatrix} \alpha\partial_1 - a_3\partial_2 & \beta\partial_1 - b_3\partial_2 \\ \alpha\partial_2 + a_3\partial_1 & \beta\partial_2 + b_3\partial_1 \end{bmatrix}, \qquad B = \begin{bmatrix} \alpha\partial_1 - \frac{\mu}{\lambda+2\mu}a_3\partial_2 & \frac{\lambda+2\mu}{\mu}\beta\partial_1 - b_3\partial_2 \\ \alpha\partial_2 + \frac{\mu}{\lambda+2\mu}a_3\partial_1 & \frac{\lambda+2\mu}{\mu}\beta\partial_2 + b_3\partial_1 \end{bmatrix}. \tag{29}$$

The determinants of (29) are equal to $|C| = |B| = (\alpha b_3 - \beta a_3)(\partial_{11} + \partial_{22})$; and also $\alpha b_3 - \beta a_3 \neq 0$.

The matrices (27)–(29) satisfy (12), (17). The factor $\operatorname{diag}(\lambda + 2\mu, \mu)$ in D in (28) can be attached to the matrix B of (29). Then the solution to (1) in the two-dimensional case with matrix (27) is as follows:

$$u_1 = (\alpha\partial_1 - a_3\partial_2)\varphi_1 + (\beta\partial_1 - b_3\partial_2)\varphi_2, \qquad u_2 = (\alpha\partial_2 + a_3\partial_1)\varphi_1 + (\beta\partial_2 + b_3\partial_1)\varphi_2, \tag{30}$$

$$(\partial_{11} + \partial_{22})\varphi_1 = f_1, \qquad (\partial_{11} + \partial_{22})\varphi_2 = f_2, \tag{31}$$

$$\begin{gathered} f_1 + [(\lambda + 2\mu)\beta\partial_1 - \mu b_3\partial_2]f_2 = 0, \\ [(\lambda + 2\mu)\alpha\partial_2 + \mu a_3\partial_1]f_1 + [(\lambda + 2\mu)\beta\partial_2 + \mu b_3\partial_1]f_2 = 0. \end{gathered} \tag{32}$$

The following formulas are well known [9] (the bar over a letter stands for complex conjugation):

$$\partial_z = (\partial_1 - i\partial_2)/2, \qquad \partial_{\bar z} = (\partial_1 + i\partial_2)/2, \qquad z = x_1 + ix_2, \qquad i = \sqrt{-1}. \tag{33}$$

Involving (33), rewrite (30) and (32):

$$\begin{aligned} u_1 + iu_2 = \partial_1[\alpha\varphi_1 + \beta\varphi_2 + i(a_3\varphi_1 + b_3\varphi_2)] + i\partial_2[\alpha\varphi_1 + \beta\varphi_2 + i(a_3\varphi_1 + b_3\varphi_2)] \\ = 2\partial_{\bar z}[\alpha\varphi_1 + \beta\varphi_2 + i(a_3\varphi_1 + b_3\varphi_2)], \end{aligned} \tag{34}$$

$$\begin{gathered} \partial_1[(\lambda + 2\mu)(\alpha f_1 + \beta f_2)] - \partial_2[\mu(a_3 f_1 + b_3 f_2)] = 0, \\ \partial_2[(\lambda + 2\mu)(\alpha f_1 + \beta f_2)] + \partial_1[\mu(a_3 f_1 + b_3 f_2)] = 0. \end{gathered} \tag{35}$$

Equations (35) are the Cauchy–Riemann conditions for the analytic function (here the prime stands for the derivative with respect to z):

$$\Phi_1'(z) = (\lambda + 2\mu)(\alpha f_1 + \beta f_2) + i\mu(a_3 f_1 + b_3 f_2).$$

The last formula implies

$$\overline{\Phi_1'(z)} = (\lambda + 2\mu)(\alpha f_1 + \beta f_2) - i\mu(a_3 f_1 + b_3 f_2),$$
$$2(\alpha f_1 + \beta f_2) = \frac{1}{\lambda + 2\mu}\left[\Phi_1'(z) + \overline{\Phi_1'(z)}\right], \tag{36}$$
$$2i(a_3 f_1 + b_3 f_2) = \frac{1}{\mu}\left[\Phi_1'(z) - \overline{\Phi_1'(z)}\right].$$

Now, from (31) and (36) we obtain

$$2(\partial_{11} + \partial_{22})(\alpha\varphi_1 + \beta\varphi_2) = 2(\alpha f_1 + \beta f_2), \qquad 2i(\partial_{11} + \partial_{22})(a_3\varphi_1 + b_3\varphi_2) = 2i(a_3 f_1 + b_3 f_2);$$

$$2(\partial_{11} + \partial_{22})[\alpha\varphi_1 + \beta\varphi_2 + i(a_3\varphi_1 + b_3\varphi_2)]$$
$$= 2[\alpha f_1 + \beta f_2 + i(a_3 f_1 + b_3 f_2)] = \frac{\lambda + 3\mu}{(\lambda + 2\mu)\mu}\,\Phi_1'(z) - \frac{\lambda + \mu}{(\lambda + 2\mu)\mu}\,\overline{\Phi_1'(z)}.$$

Since $\partial_{11} + \partial_{22} = 4\partial_z\partial_{\bar z}$, from the last relation we find

$$2\partial_z\partial_{\bar z}[\alpha\varphi_1 + \beta\varphi_2 + i(a_3\varphi_1 + b_3\varphi_2)] = \frac{\lambda + 3\mu}{4(\lambda + 2\mu)\mu}\Phi_1'(z) - \frac{\lambda + \mu}{4(\lambda + 2\mu)\mu}\,\overline{\Phi_1'(z)},$$
$$2\partial_{\bar z}[\alpha\varphi_1 + \beta\varphi_2 + i(a_3\varphi_1 + b_3\varphi_2)] = \frac{\lambda + 3\mu}{4(\lambda + 2\mu)\mu}\Phi_1(z) - \frac{\lambda + \mu}{4(\lambda + 2\mu)\mu} z\overline{\Phi_1'(z)} - \overline{\psi(z)}, \tag{37}$$

where $\psi(z)$ is an analytic function that has appeared after integration over z. Put

$$\frac{\lambda + \mu}{4(\lambda + 2\mu)\mu}\Phi_1(z) = \varphi(z),$$

Then from (34) and (37) we infer

$$u_1 + iu_2 = \varkappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)}, \qquad \varkappa = \frac{\lambda + 3\mu}{\lambda + \mu}. \tag{38}$$

Thus, in the two-dimensional case, the general solution (30)–(32) implies the well-known Kolosov–Muskhelishvili formula (38) [9] which gives a representation for the displacements via two analytic functions $\varphi(z)$ and $\psi(z)$ of the complex variable $z = x_1 + ix_2$.

Formula (38) is effectively applied to solving two-dimensional boundary value problems of elasticity theory [9]. Nevertheless, in considering boundary value problems, we can also directly use solution (30)–(32), and in the space problems, the general representations (3), (15), and (16). The presence of free parameters in these representations makes it possible to simplify the boundary conditions.

Granted (29), in the two-dimensional case, we also find the symmetry (recursion) operator $Q = CB'$ [5, 7]; here

$$Q_{11} = (\lambda + 2\mu)(\alpha^2 + \beta^2)\partial_{11} - (\lambda + 3\mu)(\alpha a_3 + \beta b_3)\partial_{12} + \mu\big(a_3^2 + b_3^2\big)\partial_{22},$$

$$Q_{21} = (\lambda + 2\mu)(\alpha a_3 + \beta b_3)\partial_{11} + \big[(\lambda + 2\mu)(\alpha^2 + \beta^2) - \mu\big(a_3^2 + b_3^2\big)\big]\partial_{12} - \mu(\alpha a_3 + \beta b_3)\partial_{22},$$

$$Q_{12} = \mu(\alpha a_3 + \beta b_3)\partial_{11} + \big[(\lambda + 2\mu)(\alpha^2 + \beta^2) - \mu\big(a_3^2 + b_3^2\big)\big]\partial_{12} - (\lambda + 2\mu)(\alpha a_3 + \beta b_3)\partial_{22},$$

$$Q_{22} = \mu\big(a_3^2 + b_3^2\big)\partial_{11} + (\lambda + 3\mu)(\alpha a_3 + \beta b_3)\partial_{12} + (\lambda + 2\mu)(\alpha^2 + \beta^2)\partial_{22}.$$

Relations (17) and (19) imply the conservation laws [7, 8]

$$C_{i1}A_{ij}u_j = D_{11}B_{j1}u_j = \partial_1(\partial_{11} + \partial_{22})[\alpha(\lambda + 2\mu)u_1 + \mu a_3 u_2]$$
$$+ \partial_2(\partial_{11} + \partial_{22})[\alpha(\lambda + 2\mu)u_2 - \mu a_3 u_1] = \partial_1 A_1 + \partial_2 A_2 = 0,$$

$$C_{i2}A_{ij}u_j = D_{22}B_{j2}u_j = \partial_1(\partial_{11} + \partial_{22})[\beta(\lambda + 2\mu)u_1 + \mu b_3 u_2]$$
$$+ \partial_2(\partial_{11} + \partial_{22})[\beta(\lambda + 2\mu)u_2 - \mu b_3 u_1] = \partial_1 B_1 + \partial_2 B_2 = 0.$$

Conservation laws can also be applied in solving boundary value problems [8].

The above-proposed method of diagonalization of the system of static equations of elasticity theory and obtaining the general solution can also be applied to other systems of equations of elliptic type. Consider, for example, two systems of equations from [10]. The matrices of the operators for these systems are as follows:

$$A = \begin{bmatrix} \partial_{11} - \partial_{22} & -2\partial_{12} \\ 2\partial_{21} & \partial_{11} - \partial_{22} \end{bmatrix}, \tag{39}$$

$$A = \begin{bmatrix} \partial_{11} - \partial_{22} & \sqrt{2}\,\partial_{12} \\ \sqrt{2}\,\partial_{21} & -\partial_{11} + \partial_{22} \end{bmatrix}. \tag{40}$$

The matrix (39) is nonsymmetric; i.e., it does not correspond to equations of elasticity theory. But (40) neither corresponds to the equations of elasticity theory for a real material. For (40), we would have the matrix of elasticity moduli c_{ij} of the form

$$c_{ij} = \begin{bmatrix} 1 & 1+\sqrt{2} & 0 \\ 1+\sqrt{2} & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}. \tag{41}$$

The matrix (41) is not positive-definite; i.e., it does not correspond to any real material.

The matrices (39) and (40) corresponds to elliptic systems [10] since

$$|A| = (\partial_{11} - \partial_{22})^2 + 4\partial_{1122} = (\partial_{11} + \partial_{22})^2 > 0,$$

$$\begin{aligned} |A| &= (\partial_{11} - \partial_{22})(-\partial_{11} + \partial_{22}) - 2\partial_{1122} = -(\partial_{1111} + \partial_{2222}) \\ &= -(\partial_{11} + \sqrt{2}\partial_{12} + \partial_{22})(\partial_{11} - \sqrt{2}\partial_{12} + \partial_{22}) < 0 \end{aligned} \tag{42}$$

for any real values of the symbols ∂_1, ∂_2, and $\partial_{11} + \partial_{22} \neq 0$. Quadratic forms in the parentheses in the last expression of (42) are positive definite.

Expressions (42) imply that the equivalent diagonal matrices D for (39) and (40) must have the form

$$D = \begin{bmatrix} \partial_{11} + \partial_{22} & 0 \\ 0 & \partial_{11} + \partial_{22} \end{bmatrix}, \tag{43}$$

$$D = \begin{bmatrix} \partial_{11} + \sqrt{2}\,\partial_{12} + \partial_{22} & 0 \\ 0 & \partial_{11} - \sqrt{2}\,\partial_{12} + \partial_{22} \end{bmatrix}. \tag{44}$$

Requiring (12) for (39) and (43), find the matrices

$$C = \begin{bmatrix} \alpha_{111}\partial_1 + \alpha_{121}\partial_2 & \alpha_{112}\partial_1 \\ \alpha_{121}\partial_1 - \alpha_{111}\partial_2 & -\alpha_{112}\partial_2 \end{bmatrix}, \qquad B = \begin{bmatrix} \alpha_{111}\partial_1 - \alpha_{121}\partial_2 & \alpha_{112}\partial_1 \\ \alpha_{121}\partial_1 + \alpha_{111}\partial_2 & \alpha_{112}\partial_2 \end{bmatrix}; \tag{45}$$

Moreover,

$$|C| = |B| = -\alpha_{121}\alpha_{112}(\partial_{11} + \partial_{22}).$$

The coefficients in (45) are free parameters but $\alpha_{121}\alpha_{112} \neq 0$. Now, with account taken of (43) and (45), the general representation of a solution to the elliptic system with operator matrix (39) is written in the form (5):

$$u_1 = (\alpha_{111}\partial_1 + \alpha_{121}\partial_2)\varphi_1 + \alpha_{112}\partial_1\varphi_2, \qquad u_2 = (\alpha_{121}\partial_1 - \alpha_{111}\partial_2)\varphi_1 - \alpha_{112}\partial_2\varphi_2, \tag{46}$$

$$(\partial_{11} + \partial_{22})\varphi_1 = f_1, \qquad (\partial_{11} + \partial_{22})\varphi_2 = f_2, \tag{47}$$

$$(\alpha_{111}\partial_1 - \alpha_{121}\partial_2)f_1 + \alpha_{112}\partial_1 f_2 = 0, \qquad (\alpha_{121}\partial_1 + \alpha_{111}\partial_2)f_1 + \alpha_{112}\partial_2 f_2 = 0. \tag{48}$$

Involving (33), rewrite (46):

$$u_1 + iu_2 = 2\partial_z[(\alpha_{111} + i\alpha_{121})\varphi_1 + \alpha_{112}\varphi_2]. \tag{49}$$

Multiplying (47) by the corresponding coefficients, we infer

$$4\partial_z\partial_{\bar z}[(\alpha_{111} + i\alpha_{121})\varphi_1 + \alpha_{112}\varphi_2] = (\alpha_{111} + i\alpha_{121})f_1 + \alpha_{112}f_2. \tag{50}$$

Equations (48) are rewritten as

$$2\partial_{\bar z}[(\alpha_{111} + i\alpha_{121})f_1 + \alpha_{112}f_2] = 0.$$

Hence, $(\alpha_{111} + i\alpha_{121})f_1 + \alpha_{112}f_2 = 2\varphi(z)$ is an analytic function. Then from (50) we obtain

$$4\partial_z\partial_{\bar z}[(\alpha_{111} + i\alpha_{121})\varphi_1 + \alpha_{112}\varphi_2] = 2\varphi(z),$$
$$2\partial_z[(\alpha_{111} + i\alpha_{121})\varphi_1 + \alpha_{112}\varphi_2] = \bar z\varphi(z) + \psi(z)$$

($\psi(z)$ is an analytic function that has appeared after integration over $\bar z$). Inserting the last expression in (49), we find

$$u_1 + iu_2 = \bar z\varphi(z) + \psi(z). \tag{51}$$

Formula (51) coincides with the solution in [10], as it should be.

Since the matrix (39) is not symmetric, (12) implies the relation $C'A' = DB'$. If $\tilde u$ is a solution to the adjoint (transposed) equations $A'\tilde u = 0$ then $u = CB'\tilde u$ is a solution to the initial equations

$$Au = ACB'\tilde u = BDB'\tilde u = BC'A'\tilde u = 0.$$

Requiring the fulfillment of (12) for (40) and (44), find the matrices

$$C = \begin{bmatrix} \alpha_{111}\partial_1 + \alpha_{121}\partial_2 & \alpha_{112}\partial_1 + \alpha_{122}\partial_2 \\ (\alpha_{111} - \sqrt{2}\alpha_{121})\partial_1 + (\sqrt{2}\alpha_{111} - \alpha_{121})\partial_2 & -(\alpha_{112} + \sqrt{2}\alpha_{122})\partial_1 + (\sqrt{2}\alpha_{112} + \alpha_{122})\partial_2 \end{bmatrix},$$
$$B = \begin{bmatrix} \alpha_{111}\partial_1 - \alpha_{121}\partial_2 & \alpha_{112}\partial_1 - \alpha_{122}\partial_2 \\ (\sqrt{2}\alpha_{121} - \alpha_{111})\partial_1 + (\sqrt{2}\alpha_{111} - \alpha_{121})\partial_2 & (\alpha_{112} + \sqrt{2}\alpha_{122})\partial_1 + (\sqrt{2}\alpha_{112} + \alpha_{122})\partial_2 \end{bmatrix}. \tag{52}$$

The determinants of the matrices (52) look as follows:

$$|C| = -|B| = (\sqrt{2}\alpha_{121}\alpha_{112} - 2\alpha_{111}\alpha_{112} - \sqrt{2}\alpha_{111}\alpha_{122})\partial_{11} + (\sqrt{2}\alpha_{121}\alpha_{112} + 2\alpha_{121}\alpha_{122} - \sqrt{2}\alpha_{111}\alpha_{122})\partial_{22}$$

and differ by sign. Free parameters in (52) must be such that the determinant $|C|$ is nonzero. Thus, granted (44) and (52), the general representation of a solution to the elliptic system with operator matrix (40) is written in the form (5):

$$u_1 = (\alpha_{111}\partial_1 + \alpha_{121}\partial_2)\varphi_1 + (\alpha_{112}\partial_1 + \alpha_{122}\partial_2)\varphi_2,$$

$$u_2 = [(\alpha_{111} - \sqrt{2}\alpha_{121})\partial_1 + (\sqrt{2}\alpha_{111} - \alpha_{121})\partial_2]\varphi_1 + [-(\alpha_{112} + \sqrt{2}\alpha_{122})\partial_1 + (\sqrt{2}\alpha_{112} + \alpha_{122})\partial_2]\varphi_2,$$

$$(\partial_{11} + \sqrt{2}\partial_{12} + \partial_{22})\varphi_1 = f_1, \qquad (\partial_{11} - \sqrt{2}\partial_{12} + \partial_{22})\varphi_2 = f_2,$$

$$(\alpha_{111}\partial_1 - \alpha_{121}\partial_2)f_1 + (\alpha_{112}\partial_1 - \alpha_{122}\partial_2)f_2 = 0,$$

$$[(\sqrt{2}\alpha_{121} - \alpha_{111})\partial_1 + (\sqrt{2}\alpha_{111} - \alpha_{121})\partial_2]f_1 + [(\alpha_{112} + \sqrt{2}\alpha_{122})\partial_1 + (\sqrt{2}\alpha_{112} + \alpha_{122})\partial_2]f_2 = 0.$$

For (40), (44), and (52), relation (17) also holds, and the expression $u = CB'\tilde u$ is the formula for producing new solutions; i.e., $Q = CB'$ is the symmetry (recursion) operator.

Thus, the method of the present article makes it possible to find a general representation for a solution not only to the equations of elasticity theory but also to other elliptic systems of equations. The method is also applicable to systems of equations with variable coefficients. Then the corresponding formulas must contain adjoint operators instead of transposed operators.

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